5 Hedging Interest-Rate Risk with Duration

Before implementing any kind of hedging method against the interest-rate risk, we need to understand how bond prices change, given a change in interest rates. This is critical to successful bond management.

5.1 Basics of Interest-Rate Risk: Qualitative Insights

The basics of bond price movements as a result of interest-rate changes are perhaps best summarized by the five theorems on the relationship between bond prices and yields. As an illustration (see Table 5.1), let us consider the percentage price change for 4 bonds with different annual coupon rates (8% and 5%) and different maturities (5 years and 25 years), starting with a common 8% yield-to-maturity (YTM), and assuming successively a new yield of 5%, 7%, 7.99%, 8.01%, 9% and 11%.

From this example, we can make the following observations. Using the bond valuation model, one can show the changes that occur in the price of a bond (i.e., its volatility), given a change in yields, as a result of bond variables such as time to maturity and coupon, and show that these observations actually hold in all generalities. For now, we simply state these "theorems." More detailed comments about these elements will follow. We leave the proof of these theorems as an exercise to the mathematically oriented reader.

Table 5.1 Percentage Price Change for 4 Bonds, Starting with a Common 8% YTM.

New yield (%)	Change (bps)	8%/25 (%)	8%/5 (%)	5%/25 (%)	5%/5 (%)
5.00	−300	42.28	12.99	47.11	13.61
7.00	−100	11.65	4.10	12.82	4.29
7.99	−1	0.11	0.04	0.12	0.04
8.01	+1	−0.11	−0.04	−0.12	−0.04
9.00	+100	−9.82	−3.89	−10.69	−4.07
11.00	+300	−25.27	−11.09	−27.22	−11.58

5.1.1 The Five Theorems of Bond Pricing

- *Bond prices move inversely to interest rates. Investors must always keep in mind a fundamental fact about the relationship between bond prices and bond yields: bond prices move inversely to market yields. When the level of required yields demanded by investors on new issues changes, the required yields on all bonds already outstanding will also change. For these yields to change, the prices of these bonds must change. This inverse relationship is the basis for understanding, valuing and managing bonds.*
- *Holding maturity constant, a decrease in rates will raise bond prices on a percentage basis more than a corresponding increase in rates will lower bond prices. Obviously, bond price volatility*

can work for, as well as against, investors. Money can be made, and lost, in risk-free Treasury securities as well as in riskier corporate bonds.

- *All things being equal, bond price volatility is an increasing function of maturity. Long-term bond prices fluctuate more than short-term bond prices. Although the inverse relationship between bond prices and interest rates is the basis of all bond analysis, a complete understanding of bond price changes as a result of interest-rate changes requires additional information. An increase in interest rates will cause bond prices to decline, but the exact amount of decline will depend on important variables unique to each bond such as time to maturity and coupon. An important principle is that for a given change in market yields, changes in bond prices are directly related to time to maturity. Therefore, as interest rates change, the prices of longer-term bonds will change more than the prices of shorter-term bonds, everything else being equal.*
- *A related principle regarding maturity is as follows: the percentage price change that occurs as a result of the direct relationship between a bond's maturity and its price volatility increases at a decreasing rate as time to maturity increases. In other words, the percentage of price change resulting from an increase in time to maturity increases, but at a decreasing rate. Put simply, a doubling of the time to maturity will not result in a doubling of the percentage price change resulting from a change in market yields.*
- *In addition to the maturity effect, the change in the price of a bond as a result of a change in interest rates depends on the coupon rate of the bond. We can state this principle as (other things equal): bond price fluctuations (volatility) and bond coupon rates are inversely related. Note that we are talking about percentage price fluctuations; this relationship does not necessarily hold if we measure volatility in terms of dollar price changes rather than percentage price changes.*

These principles lead to the practical conclusion that the two bond variables of major importance in assessing the change in the price of a bond, given a change in interest rates, are its coupon and its maturity. This conclusion can be summarized as follows: A decline (rise) in interest rates will cause a rise (decline) in bond prices, with the maximum volatility in bond prices occurring in longer maturity bonds and in bonds with low coupons. Therefore, a bond buyer, in order to receive the maximum price impact of an expected change in interest rates, should purchase low-coupon, long-maturity bonds. If an increase in interest rates is expected (or feared), investors contemplating their purchase should consider those bonds with large coupons or short maturities, or both.

These relationships provide useful information for bond investors by demonstrating how the price of a bond changes as interest rates change. Although investors have no control over the change and direction in market rates, they can exercise control over the coupon and maturity, both of which have significant effects on bond price changes.

An important distinction needs to be made between two kinds of risk, reinvestment risk and capital gain risk.

5.1.2 Reinvestment Risk

It is important to understand that the YTM is a promised yield, because investors earn the indicated yield only if the bond is held to maturity and the coupons are reinvested at the calculated YTM (yield to maturity). Obviously, no trading can be done for a particular bond if the YTM is to be earned. The investor simply buys and holds. What is not so obvious to many investors, however, is the reinvestment implications of the YTM measure. Because of the importance of the reinvestment rate, we consider it in more detail by analyzing the reinvestment risk.

The YTM calculation assumes that the investor reinvests all coupons received from a bond at a rate equal to the computed YTM on that bond, thereby earning interest on interest over the life of the bond at the computed YTM rate.

If the investor spends the coupons, or reinvests them at a rate different from the YTM, the realized yield that will actually be earned at the termination of the investment in the bond will differ from the promised YTM. In fact, coupons almost always will be reinvested at rates higher or lower than the computed YTM, resulting in a realized yield that differs from the promised yield. This gives rise to reinvestment rate risk.

This interest-on-interest concept significantly affects the potential total dollar return. The exact impact is a function of coupon and time to maturity, with reinvestment becoming more important as either coupon or time to maturity, or both, rise. Specifically,

- *holding everything else constant, the longer the maturity of a bond, the greater the reinvestment risk;*
- *holding everything else constant, the higher the coupon rate, the greater the dependence of the total dollar return from the bond on the reinvestment of the coupon payments.*

Lets look at realized yields under different assumed reinvestment rates for a 20-year bullet bond purchased at a $100 face value, which delivers an annual 10% coupon rate. If the reinvestment rate exactly equals 10% YTM, the investor would realize a 10% compound return when the bond is held to maturity, with $372.75 of the total dollar return from the bond attributable to the reinvestment of the coupon payments. At a 12% reinvestment rate, the investor would realize an 11.10% compound return, with 63.4% of the total return coming from interest on interest ($520.52/$820.52). With no reinvestment of coupons (spending them as received), the investor would achieve only a 5.65% return. In all cases, the bond is held to maturity.

Clearly, the reinvestment portion of the YTM concept is critical. In fact, for long-term bonds the interest-on-interest component of the total realized yield may account for more than three-fourths of the bond's total dollar return.

5.1.3 Capital Gain Risk

As bond yield drops, bond price rises, and vice versa.

Example 5.1 Consider a 2-year, 10% coupon bond with a $1,000 face value, and a yield of 8.8%, which pays a semiannual coupon. The price of the bond equals $1,021.58:

$$\frac{50}{(1+4.4\%)}+\frac{50}{(1+4.4\%)^2}+\frac{50}{(1+4.4\%)^3}+\frac{1{,}050}{(1+4.4\%)^4}=\$1{,}021.58$$

Suppose the market interest rate drops instantaneously to 7.8%. The market price is now \$1,040.02:

$$\frac{50}{(1+3.9\%)}+\frac{50}{(1+3.9\%)^2}+\frac{50}{(1+3.9\%)^3}+\frac{1{,}050}{(1+3.9\%)^4}=\$1{,}040.02$$

There is an interplay between capital gain risk and reinvestment risk.

Example 5.2 An investor with a horizon of 1 year buys the 2-year, 10% coupon bond of the previous example. The bond yield is 8%.

- ***Scenario 1:*** *The interest rate drops to 7.8%, soon after the bond is purchased, and stays there. The bond price rises immediately to 1,040.02. After 6 months, the bond price rises to* $1{,}040.02 \times 1.039 = 1{,}080.58$*. At this time, a coupon of \$50 is paid, whereupon the price of the bond drops by an equal amount to 1,030.58. The bond increases in value at the end of the year to* $1{,}030.58 \times 1.039 = 1{,}070.78$*, while the \$50 coupon has been reinvested at an annual yield of 7.8% and has grown to* $50 \times 1.039 = 51.95$*, for a total of 1,122.73.*
- ***Scenario 2:*** *The interest rate rises to 8.2%, and stays there. The price immediately drops to 1,032.59. However, the bond and coupons thereafter rise at the rate of 4.1% every 6 months, culminating in a value of 1,119.00 at the end of the investment horizon.*

What we can see is that if interest rates drop and stay there, there is an immediate appreciation in the value of the portfolio, but the portfolio then grows at a slower rate; on the other hand, if interest rates rise and stay there, there is a capital loss, but the portfolio then appreciates more rapidly. Hence, there is some investment horizon D, such that investors with that horizon will not care if interest rates drop or rise (as long as the changes are small). We will see below that the value of this horizon depends on the characteristics of the bond portfolio; specifically, we will see that D is simply the duration of the bond portfolio.

Despite the usefulness of the qualitative insights obtained above, quantitative tools must be introduced to provide investors with more definite answers to the most important question they face: what is the dollar impact of a given change in market conditions on the value of my bond portfolio? That is why we have to first qualify the interest-rate risk.

5.1.4 Qualifying Interest-Rate Risk

A portfolio manager aims at hedging the value of a fixed-income portfolio that delivers certain (or deterministic) cash flows in the future, typically cash flows from straight bonds with a fixed-coupon rate. Even if these cash flows are known in advance, the portfolio price changes in time, which leaves an investor exposed to a potentially significant capital loss.

To fix the notation, we consider at date t a portfolio of fixed-income securities that delivers m certain cash flows F_i at future dates t_i for $i = 1, \ldots, m$. The price P of the portfolio (in \$ value) can be written as the sum of the future cash flows discounted with the appropriate zero-coupon rate with maturity corresponding to the maturity of each cash flow:

$$P_t = \sum_{i=1}^{m} F_i B(t, t_i) = \sum_{i=1}^{m} \frac{F_i}{[1 + R(t, t_i - t)]^{t_i - t}} \tag{5.1}$$

where $B(t, t_i)$ is the price at date t of a zero-coupon bond paying \$1 at date t_i (also called the *discount factor*) and $R(t, t_i - t)$ is the associated zero-coupon rate, starting at date t for a residual maturity of $t_i - t$ years.

We can see in equation (5.1) that the price P_t is a function of m interest-rate variables $R(t, t_i - t)$ and of the time variable t. This suggests that the value of the portfolio is subject to a potentially large number m of risk factors. For example, a bond with annual cash flows up to a 10-year maturity is affected by potential changes in 10 zero-coupon rates. To hedge a position in this bond, we need to be hedged against a change in all of these 10 factor risks.

In practice, it is not easy to hedge the risk of so many variables. We must create a global portfolio containing the portfolio to be hedged in such a way that the portfolio is insensitive to all sources of risk (the m interest-rate variables and the time variable t).[1] One suitable way to simplify the hedging problematic is to reduce the number of risk variables. Duration hedging of a portfolio is based on a single risk variable, the yield to maturity of this portfolio.

5.2 Hedging with Duration

The whole idea behind duration hedging is to bypass the complication of a multidimensional interest-rate risk by identifying a single risk factor, the yield to maturity of the portfolio, which will serve as a "proxy" for the whole term structure. We study the sensitivity of the price of the bond to changes in this yield using a one-order Taylor expansion.

5.2.1 Using a One-Order Taylor Expansion

The first step consists in writing the price of the portfolio P_t in \$ value as a function of a single source of interest-rate risk, its yield to maturity y_t (see equation (5.2)):

$$P_t = P(y_t) = \sum_{i=1}^{m} \frac{F_i}{[1 + y_t]^{t_i - t}} \tag{5.2}$$

Remark 5.1 For semiannual coupon-bearing bonds, the formula above should be expressed as

$$P_t = \sum_{i=1}^{m} \frac{F_i}{\left(1 + \frac{y_t}{2}\right)^{2(t_i - t)}} \tag{5.3}$$

where y_t is a semiannual compounded yield and $t_i - t$ a fraction of the year before cash flow i is paid. As an example, we consider a \$1,000 face value 2-year bond with 8% coupon and

[1] Hereafter, we do not consider the change in value due to time because it is a deterministic term [for details about the time-value of a bond, see Chance and Jordan (1996)]. We only consider changes in value due to interest-rate variations.

a yield to maturity $y = 6\%$. The bond price is obtained as

$$P = \frac{40}{1+\frac{y}{2}} + \frac{40}{\left(1+\frac{y}{2}\right)^2} + \frac{40}{\left(1+\frac{y}{2}\right)^3} + \frac{1{,}040}{\left(1+\frac{y}{2}\right)^4} = \$1{,}037.17$$

Therefore, equation (5.3) should rather be written as

$$P_t = \sum_{i=1}^{m} \frac{F_i}{(1+y'_t)^{\theta_i}}$$

with the convention $y'_t = \frac{y_t}{2}$, and $\theta_i = 2(t_i - t)$ being a number of half-years. For notational simplicity, we choose to maintain equation (5.2) instead with y and $t_i - t$. We hope not to confuse the reader by doing so.

In this case, we can see clearly that the interest-rate risk is (imperfectly) summarized by changes in the yield to maturity y_t. Of course, this can only be achieved by losing much generality and imposing important, rather arbitrary and simplifying assumptions. The yield to maturity is a complex average of the whole term structure, and it can be regarded as the term structure if and only if the term structure is flat.

A second step involves the derivation of a Taylor expansion of the value of the portfolio P as an attempt to quantify the magnitude of value changes $\mathrm{d}P$ that are triggered by small changes $\mathrm{d}y$ in yield. We get an approximation of the *absolute* change in the value of the portfolio as

$$\mathrm{d}P(y) = P(y+\mathrm{d}y) - P(y) = P'(y)\,\mathrm{d}y + o(y) \simeq \$Dur(P(y))\,\mathrm{d}y \tag{5.4}$$

where

$$P'(y) = -\sum_{i=1}^{m} \frac{(t_i - t)F_i}{[1+y_t]^{t_i - t + 1}}$$

The derivative of the bond value function with respect to the yield to maturity is known as the *\$duration* (or *sensitivity*) of portfolio P, and $o(y)$ a negligible term. From equation (5.4), we confirm an important result discussed in the qualitative approach to the problem: $\$Dur < 0$. What this means is that the relationship between price and yield is negative; higher yields imply lower prices.

Dividing equation (5.4) by $P(y)$, we obtain an approximation of the *relative* change in value of the portfolio as

$$\frac{\mathrm{d}P(y)}{P(y)} = \frac{P'(y)}{P(y)}\mathrm{d}y + o_1(y) \simeq -MD(P(y))\,\mathrm{d}y \tag{5.5}$$

where

$$MD(P(y)) = -\frac{P'(y)}{P(y)}$$

is known as the *modified duration (MD)*[2] of portfolio P.

[2] Note that the opposite quantity of the modified duration is known as the *relative sensitivity*.

The \$duration and the modified duration enable us to compute the absolute P&L and the relative P&L of portfolio P for a small change Δy of the yield to maturity (for example, 10 bps or 0.1% as expressed in percentage):

$$\text{Absolute } P\&L \simeq \$Dur \times \Delta y$$

$$\text{Relative } P\&L \simeq -MD \times \Delta y$$

The \$duration and the modified duration are also measures of the volatility of a bond portfolio.

Another standard measure is the basis point value (BPV), which is the change in the bond price given a basis point change in the bond's yield. BPV is given by the following equation:

$$BPV = \frac{MD \times P}{10{,}000} = \frac{-\$Dur}{10{,}000}$$

BPV is typically used for hedging bond positions.

Example 5.3 \$Duration, Modified Duration and Basis Point Value

Consider below a bond with the following features:

- ***maturity:*** *10 years*
- ***coupon rate:*** *6%*
- ***YTM:*** *5% (or price: \$107.72)*

Coupon frequency and compounding frequency are assumed to be annual.

The \$duration of this bond is equal to −809.67, the BPV is 0.0809 and the modified duration is equal to 7.52. If the YTM increases by 0.1%, the holder of the bond will incur in first approximation an absolute loss equal to

$$\textit{Absolute Loss} = -809.67 \times 0.1\% = -\$0.80967$$

Its relative loss expressed in percentage is

$$\textit{Relative Loss} = -7.52 \times 0.1\% = -0.752\%$$

Remark 5.2 \$Duration and Modified Duration of a Bond with Semiannual Payments

Considering the price of a bond with semiannual payments given by equation (5.3), we obtain the following expression for $P'(y)$, which is the derivative of the bond price with respect to the yield to maturity y

$$P'(y) = -\sum_{i=1}^{m} \frac{(t_i - t)F_i}{\left[1 + \frac{y_t}{2}\right]^{2(t_i - t)+1}}$$

We now consider the same bond as in the previous example (maturity = 10 years; coupon rate = 6%; YTM = 5%) except that coupon frequency and compounding frequency are now assumed to be semiannual.

The price, $duration and modified duration of this bond are equal to $107.79, −816.27 and 7.57, respectively.

5.2.2 Duration, $Duration and Modified Duration

Different Notions of Duration

There are three different notions of duration. We already have seen two of them, the $duration and the modified duration, which are used to compute the absolute P&L and the relative P&L of the bond portfolio for a small change in the yield to maturity. The third one is the Macaulay duration, simply called *duration*. The duration of bond P is defined as

$$D \equiv D(P(y)) \equiv -(1+y)\frac{P'(y)}{P(y)} = \frac{\sum_{i=1}^{m} \frac{(t_i - t)F_i}{[1+y]^{t_i - t}}}{P(y)} \tag{5.6}$$

The duration may be interpreted as a weighted average maturity for the portfolio. The weighted coefficient for each maturity $t_i - t$ is equal to

$$w = \frac{F_i}{P(y)(1+y)^{t_i - t}}$$

It is easy to check that the duration is always less than or equal to the maturity, and is equal to it if and only if the portfolio has only one cash flow (e.g., zero-coupon bond). Duration is very easy to compute in practice, as can be seen through the following example.

Example 5.4 Calculation of Bond Duration

We consider a bond paying 10 annual cash flows, with a coupon rate $c = 5.34\%$ and a yield to maturity $y = 5.34\%$ (with annual compounding frequency). We compute the duration as $D = 8$ (see Table 5.2). This tells us that the cash flow–weighted average maturity is 8 years. In other words, this 10-year coupon bond essentially behaves, in terms of sensitivity to interest-rate changes, as an 8-year pure discount bond.

The duration of a bond or bond portfolio is the investment horizon such that investors with that horizon will not care if interest rates drop or rise as long as changes are small. In other words, capital gain risk is offset by reinvestment risk as is shown now in the following example.

Example 5.5 Duration = Neutral Investment Horizon

Consider a 3-year standard bond with a 5% YTM and a $100 face value, which delivers a 5% coupon rate. Coupon frequency and compounding frequency are assumed to be annual. Its

Table 5.2 Calculation of Bond Duration.

Time of cash flow (t)	Cash flow F_t	$w_t = \frac{l}{P} \times \frac{F_t}{(l+y)^t}$	$t \times w_t$
1	53.4	0.0506930	0.0506930
2	53.4	0.0481232	0.0962464
3	53.4	0.0456837	0.1370511
4	53.4	0.0433679	0.1734714
5	53.4	0.0411694	0.2058471
6	53.4	0.0390824	0.2344945
7	53.4	0.0371012	0.2597085
8	53.4	0.0352204	0.2817635
9	53.4	0.0334350	0.3009151
10	1053.4	0.6261237	6.2612374
		Total	8.0014280

$$\Rightarrow D = \sum_{t=1} t \times w_t \cong 8$$

price is \$100 and its duration is equal to 2.86. We assume that YTM changes instantaneously and stays at this level during the life of the bond. Whatever the change in this YTM, we show in the following table that the sum of the bond price and the reinvested coupons after 2.86 years is always the same, equal to 114.972 or 114.971.

YTM (%)	Bond price	Reinvested coupons	Total
4	104.422	10.550	114.972
4.5	104.352	10.619	114.971
5	104.282	10.689	114.971
5.5	104.212	10.759	114.971
6	104.143	10.829	114.972

Relationships between the Different Duration Measures

Note that there is a set of simple relationships between the three different durations:

$$MD = \frac{D}{1+y}$$

$$\$Dur = -\frac{D}{1+y} \times P = -MD \times P$$

Remark 5.3 When the coupon frequency and the compounding frequency of a bond are assumed to be semiannual, these two relationships are affected in the following manner:

$$MD = \frac{D}{1+\frac{y}{2}}$$

$$\$Dur = -\frac{D}{1+\frac{y}{2}} \times P = -MD \times P$$

Properties of the Different Duration Measures

The main properties of duration, modified duration and $duration measures are as follows:

1. *The duration of a zero-coupon bond equals its time to maturity.*
2. *Holding the maturity and the YTM of a bond constant, the bond's duration (modified duration or $duration) is higher when the coupon rate is lower.*
3. *Holding the coupon rate and the YTM of a bond constant, its duration (or modified duration) increases with its time to maturity as $duration decreases.*
4. *Holding other factors constant, the duration (or modified duration) of a coupon bond is higher as $duration is lower when the bond's YTM is lower.*

Example 5.6 To illustrate points (2), (3) and (4), we consider a base-case bond with maturity 10 years, a 6% annual coupon rate, a 6% YTM (with annual compounding frequency) and a $100 face value.

Holding the maturity and the YTM of this bond constant, we give different values to the coupon rate and can see clearly that the lower the coupon rate, the higher the duration (modified duration and $duration).

Coupon rate (%)	Duration	$Duration	Modified duration
3	8.59	−631.40	8.10
4	8.28	−666.27	7.81
5	8.02	−701.14	7.57
6	7.80	−736.01	7.36
7	7.61	−770.88	7.18
8	7.45	−805.75	7.02
9	7.30	−840.62	6.89

Example 5.7 Holding the coupon rate and the YTM of this bond constant, we give different values to the maturity and can see that clearly the higher the maturity, the higher the duration (and modified duration), and the lower the $duration.

Maturity (years)	Duration	$Duration	Modified duration
7	5.92	−558.24	5.58
8	6.58	−620.98	6.21
9	7.21	−680.17	6.80
10	7.80	−736.01	7.36
11	8.36	−788.69	7.89
12	8.89	−838.38	8.38
13	9.38	−885.27	8.85

Holding the maturity and the coupon rate of this bond constant, we give different values to the YTM and can see clearly that the lower the YTM, the higher the duration (and modified duration), and the lower the $duration.

YTM (%)	Duration	$Duration	Modified duration
3	8.07	−983.62	7.83
4	7.98	−891.84	7.67
5	7.89	−809.67	7.52
6	7.80	−736.01	7.36
7	7.71	−669.89	7.20
8	7.62	−610.48	7.05
9	7.52	−557.02	6.90

5. *The duration of a perpetual bond that delivers an annual coupon c over an unlimited horizon and with a YTM equal to y is* $(1+y)/y$.

6. *Another convenient property of duration is that it is a linear operator. In other words, the duration of a portfolio P invested in n bonds denominated in the same currency with weights* w_i *is the weighted average of each bond's duration:*

$$D_p = \sum_{i=1}^{n} w_i D_i \tag{5.7}$$

A proof of these two last propositions can be found in the Exercise part.

Remark 5.4 Note that this property of linearity, also true for modified duration, is only true in the context of a flat curve. When the YTM curve is no longer flat, this property becomes false and may only be used as an approximation of the true duration (or modified duration).

5.2.3 How to Hedge in Practice?

We attempt to hedge a bond portfolio with yield to maturity y and price (in $ value) denoted by $P(y)$. The idea is to consider one hedging asset with yield to maturity y_1 (*a priori* different from y), whose price is denoted by $H(y_1)$, and to build a global portfolio with value P^* invested in the initial portfolio and some quantity ϕ of the hedging instrument.

$$P^* = P(y) + \phi H(y_1)$$

The goal is to make the global portfolio insensitive to small interest-rate variations. Using equation (5.4) and assuming that the YTM curve is only affected by parallel shifts so that $\mathrm{d}y = \mathrm{d}y_1$, we obtain

$$\mathrm{d}P^* \simeq [P'(y) + \phi H'(y_1)]\,\mathrm{d}y = 0$$

which translates into

$$\phi \$Dur(H(y_1)) = -\$Dur(P(y))$$

or

$$\phi H(y_1)MD(H(y_1)) = -P(y)MD(P(y))$$

and we finally get

$$\phi = -\frac{\$Dur(P(y))}{\$Dur(H(y_1))} = -\frac{P(y)MD(P(y))}{H(y_1)MD(H(y_1))} \tag{5.8}$$

The optimal amount invested in the hedging asset is simply equal to the opposite of the ratio of the \$duration of the bond portfolio to be hedged by the \$duration of the hedging instrument. Recall that the hedge requires taking an opposite position in the hedging instrument. The idea is that any loss (gain) with the bond has to be offset by a gain (loss) with the hedging instrument.

Remark 5.5 When the yield curve is flat, which means that $y = y_1$, we can also use the Macaulay Duration to construct the hedge of the instrument. In this particular case, the hedge ratio (HR) ϕ given by equation (5.8) is also equal to

$$\phi = -\frac{P(y)D(P(y))}{H(y)D(H(y))}$$

where the Macaulay duration $D(P(y))$ is defined in equation (5.6).

In practice, it is preferable to use futures contracts or swaps instead of bonds to hedge a bond portfolio because of significantly lower costs and higher liquidity. Using standard swaps as hedging instruments[3], the HR ϕ_s is

$$\phi_s = -\frac{\$Dur_P}{\$Dur_S} \tag{5.9}$$

where $\$Dur_S$ is the \$duration of the fixed-coupon bond contained in the swap.

Using futures as hedging instruments[4], the HR ϕ_f is equal to

$$\phi_f = -\frac{\$Dur_P}{\$Dur_{CTD}} \times CF \tag{5.10}$$

where $\$Dur_{CTD}$ is the \$duration of the cheapest-to-deliver bond, and CF the conversion factor.

We will see practical examples of hedges using swaps and futures in Chapters 10 and 11.

[3]See Chapter 10 for a complete description of interest-rate risk hedging methods with swaps.
[4]See Chapter 11 for a complete description of interest-rate risk hedging methods with futures.

Example 5.8 Hedging a Bond Position

An investor holds a bond portfolio, whose features are summarized in the following table, and wishes to be hedged against a rise in interest rates.

YTM	MD	Price
5.143%	6.760	$328,635

The characteristics of the hedging instrument, which is a bond here, are the following:

YTM	MD	Price
4.779%	5.486	$118.786

We then obtain the quantity ϕ of the hedging instrument using equation (5.8):

$$\phi = -\frac{328{,}635 \times 6.760}{118.786 \times 5.486} = -3{,}409$$

The investor has to sell 3,409 units of the hedging instrument.

Another measure commonly used by market participants to hedge their bond positions is the basis point value (BPV) measure.

Example 5.9 Hedging a Bond Position Using BPV

We calculate the hedge ratio denoted as HR, which gives the size of the hedge position

$$HR = \frac{BPV_{\mathrm{b}}}{BPV_{\mathrm{h}}} \times \frac{\textit{Change in bond yield}}{\textit{Change in yield for the hedging instrument}}$$

where BPV_{b} and BPV_{h} are, respectively, the basis point value of the bond to be hedged and of the hedging instrument. The second ratio in the equation is sometimes called the *yield ratio*.

For example, if $BPV_{\mathrm{b}} = 0.0809$ and $BPV_{\mathrm{h}} = 0.05$ and if we assume that the YTM curve is affected by a parallel shift of the same magnitude, which implies that the yield ratio is equal to 1, then the HR is 1.618. For a $10,000 long position in the bond we have to take a $16,180 short position in the hedging instrument.

5.3 End of Chapter Summary

A decline (rise) in interest rates will cause a rise (decline) in bond prices, with the most volatility in bond prices occurring in longer maturity bonds and bonds with low coupons. As a stock risk is usually proxied by its beta, which is a measure of the stock sensitivity to market movements, bond price risk is most often measured in terms of the bond interest-rate sensitivity, or duration.

This is a one-dimensional measure of the bond's sensitivity to interest-rate movements. There are actually three related notions of duration: Macaulay duration, \$duration and modified duration. Macaulay duration, often simply called *duration*, is defined as a weighted average maturity for the portfolio. The duration of a bond or bond portfolio is the investment horizon such that investors with that horizon will not care if interest rates drop or rise as long as changes are small, as capital gain risk is offset by reinvestment risk on the period. \$duration and modified duration are used to compute, respectively, the absolute P&L and the relative P&L of a bond portfolio for a small change in the yield to maturity. \$duration also provides us with a convenient hedging strategy: to offset the risks related to a small change in the level of the yield curve, one should optimally invest in a hedging asset a proportion equal to the opposite of the ratio of the \$duration of the bond portfolio to be hedged by the \$duration of the hedging instrument.

5.4 References and Further Reading

5.4.1 Books

Bierwag, G.O., 1987, *Duration Analysis: Managing Interest Rate Risk*, Ballinger Publishing Company, Cambridge, MA.

Chambers, D.R., and S.K. Nawalkha (Editors), 1999, *Interest Rate Risk Measurement and Management*, Institutional Investor, New York.

Fabozzi, F.J., 1996, *Fixed-Income Mathematics*, 3rd Edition, McGraw-Hill, New York.

Fabozzi, F.J., 1999, *Duration, Convexity and Other Bond Risk Measures*, John Wiley & Sons, Chichester.

Martellini, L., and P. Priaulet, 2000, *Fixed-Income Securities: Dynamic Methods for Interest Rate Risk Pricing and Hedging*, John Wiley & Sons, Chichester.

Macaulay, F.R., 1938, *The Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1859*, Columbia University Press, NBER, New York.

5.4.2 Papers

Bierwag, G.O., 1977, "Immunization, Duration and the Term Structure of Interest Rates", *Journal of Financial and Quantitative Analysis*, **12**, 725–742.

Bierwag, G.O., G.G. Kaufman, and A. Toevs, 1983, "Duration: Its Development and Use in Bond Portfolio Management", *Financial Analysts Journal*, **39**, 15–35.

Chance D.M., and J.V. Jordan, 1996, "Duration, Convexity, and Time as Components of Bond Returns", *Journal of Fixed Income*, **6**(2), 88–96.

Christensen P.O., and B.G. Sorensen, 1994, "Duration, Convexity and Time Value", *Journal of Portfolio Management*, **20**(2), 51–60.

Fama, E.F., and K.R. French, 1992, "The Cross-Section of Expected Stock Returns", *Journal of Finance*, **47**(2), 427–465.

Grove, M.A., 1974, "On Duration and the Optimal Maturity Structure of the Balance Sheet", *Bell Journal of Economics and Management Science*, **5**, 696–709.

Ilmanen, A., 1996, "Does Duration Extension Enhance Long-Term Expected Returns?" *Journal of Fixed Income*, **6**(2), 23–36.

Ingersoll, J.E., J. Skelton, and R.L. Weil, 1978, "Duration Forty Years After", *Journal of Financial and Quantitative Analysis*, **34**, 627–648.

Litterman, R., and J. Scheinkman, 1991, "Common Factors Affecting Bond Returns", *Journal of Fixed Income*, **1**(1), 54–61.